hep-th/9412126

# Lie group weight multiplicities from conformal field theory

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## Abstract

Dominant weight multiplicities of simple Lie groups are expressed in terms of the modular matrices of Wess-Zumino-Witten conformal field theories, and related objects. Symmetries of the modular matrices give rise to new relations among multiplicities. At least for some Lie groups, these new relations are strong enough to completely fix all multiplicities.

Short title: Weights from conformal field theory
PACS: 0220, 1110

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## 1. Introduction

Wess-Zumino-Witten (WZW) models form an important class of conformal field theories (see [1], for example). They realise a current algebra equivalent to an affine Kac-Moody algebra. Included as a subalgebra is a semi-simple Lie algebra, the global symmetry algebra of the theory. Not surprisingly then, the theory of compact Lie groups and their Lie algebras has been very useful in elucidating the properties of WZW models.

Here we present an example where the reverse is true: WZW conformal field theories tell us something useful about Lie groups and their algebras. Weight multiplicities are crucial numbers in the representation theory of Lie groups (see [2][3], for example). We will express the dominant weight multiplicities of unitary highest weight representations of Lie groups in terms of matrices relevant to WZW models. We also show that our expressions give rise to new relations among the multiplicities.

In order to explain our results, we must first discuss the connection between modular transformations and the Lie symmetry algebra of a WZW model. Primary fields of WZW models are in one-to-one correspondence with unitary highest weight representations of affine Kac-Moody algebras $\hat{g}$ [4][5]. The partition function of a WZW model on a torus is a sesquilinear combination of characters of affine Kac-Moody representations [5]. These characters transform among themselves under the action of the modular group of the torus [6]. Remarkably, ratios of elements of the modular $S$ matrix equal characters of the Lie symmetry algebra $g$ evaluated at special points (see equation (3.5) below). This happy fact, discovered by Kac and Peterson [6], is the starting point of our work, and has been much exploited elsewhere ([7][8][9], for example).

Perhaps the relation between modular matrices and semi-simple Lie algebra characters is not so surprising, if other facts are taken into account. Consider the number of couplings between three primary fields of a WZW model, the so-called fusion coefficient. Since a WZW model has a Lie symmetry algebra, this fusion coefficient is less than or equal to the corresponding tensor product coefficient of the Lie algebra [5][10]. Products of Lie algebra characters decompose into integer linear combinations of characters, according to these coefficients. E. Verlinde realised that the modular transformation matrix $S$ of a conformal field theory could tell us the fusion coefficients: products of certain ratios of elements of the modular matrix $S$ decompose into integer linear combinations of these same ratios, with the coefficients being the fusion coefficients [11]. It is reasonable that the only way such ratios could satisfy these properties is for them to coincide with the Lie algebra characters,

but evaluated at special points.

The outline of this paper is as follows. In section 2 the work of Patera and Sharp [12] is reviewed, so that it can be adapted to the use of Kac-Peterson modular matrices in section 3. There we write a new expression for the dominant weight multiplicities of semi-simple Lie algebras. The symmetries of the Kac-Peterson modular $S$ matrix, and the even Weyl sums $E^{(n)}$ we introduce in section 3, are written down in section 4. The relations between the multiplicities that follow are also given. Section 5 contains some simple explicit examples of the new relations among multiplicities, and section 6 is our conclusion.

## 2. Lie group multiplicities from Lie characters

Define the Weyl orbit sum

$$E_\lambda(\sigma) \ := \ \frac{|W(\lambda-\rho)|}{|W|} \sum_{w\in W} \sigma^{w(\lambda-\rho)} \quad . \tag{2.1}$$

Here $E$ stands for an even Weyl sum, $|W|$ is the order of the Weyl group of $g$, and $|W\lambda|$ is the order of the Weyl orbit of $\lambda\in P_+$. The notation is $\sigma^\mu = \exp[-i\mu\cdot\sigma]$.

An odd Weyl sum is the so-called discriminant

$$O_\lambda(\sigma) \ := \ \sum_{w\in W} det(w)\, \sigma^{w\lambda} \quad , \tag{2.2}$$

where $det(w)$ is the sign of the Weyl group element $w$. The Weyl character formula is

$$\chi_\lambda(\sigma) \ = \ O_\lambda(\sigma)/O_\rho(\sigma) \quad . \tag{2.3}$$

The character, being an even Weyl function, can be expanded in terms of the even functions $E_\mu$ [13]:

$$\chi_\lambda(\sigma) \ = \ \sum_{\mu\in P_{++}} \ m_\lambda^{\ \mu} E_\mu(\sigma) \quad . \tag{2.4}$$

The non-negative integers $m_\lambda^{\ \mu}$ are the dominant weight multiplicities: $m_\lambda^{\ \mu}$ denotes the multiplicity of the weight $\mu-\rho$ in the representation with highest weight $\lambda-\rho$. Note that $m_\lambda^{\ \lambda}=1$ for all $\lambda\in P_{++}$. We can consider the $m_\lambda^{\ \mu}$ to be the elements of an infinite matrix $M$. If $\lambda-\mu\notin \mathbf{Z}_\geq\{\alpha_1,\ldots,\alpha_r\}$, where $\alpha_i$ are the simple roots, then $m_\lambda^{\ \mu}=0$. Therefore $M$ is a lower triangular matrix, provided the weights are ordered appropriately.

Not only can the Weyl character be expanded in terms of the Weyl orbit sums $E_\lambda$, but the reverse is also true:

$$E_\lambda(\sigma) \;=\; \sum_{\mu\in P_{++}} \ell_\lambda{}^\mu \chi_\mu(\sigma) \quad . \tag{2.5}$$

The coefficients $\ell_\lambda{}^\mu$ are easily shown to be integers, but are *not* in general non-negative. However if $L$ is the matrix with elements $\ell_\lambda{}^\mu$, then clearly $M = L^{-1}$. So, if the triangular matrix $L$ can be calculated, it is a simple matter to invert it to obtain the dominant weight multiplicities [12].

Patera and Sharp [12] also point out that the equations above allow the calculation of the $\ell_\lambda{}^\mu$ for fixed $\lambda$ using the Weyl group. This corresponds to the following formula,

$$\ell_\lambda{}^\mu = \frac{|W(\lambda-\rho)|}{|W|} \sum_{w,x\in W} det(w)\, \delta^\mu_{w\rho+x(\lambda-\rho)} \quad , \tag{2.6}$$

that can be derived from the defining relation for the $\ell$.

## 3. Lie group multiplicities from WZW modular matrices

Define

$$O_\lambda^{(n)}(\sigma) \;:=\; F_n \sum_{w\in W} det(w)\, \sigma_{(n)}^{w\lambda} \;=\; S^{(n)}_{\lambda\sigma} \quad , \tag{3.1}$$

and

$$E_\lambda^{(n)}(\sigma) \;:=\; \frac{|W(\lambda-\rho)|}{|W|} \sum_{w\in W} \sigma_{(n)}^{w(\lambda-\rho)} \quad , \tag{3.2}$$

with

$$\sigma^\mu_{(n)} \;:=\; \exp[-2\pi i \mu\cdot\sigma/n] \;, \qquad F_n \;:=\; \frac{i^{|\Delta_+|}}{n^{r/2}\sqrt{|M^*/M|}} \quad ; \tag{3.3}$$

where $|\Delta_+|$ is the number of positive roots of $g$, and $M$ here is the weight lattice. The matrix $S^{(n)}$ in eq.(3.1) is the Kac-Peterson modular matrix of WZW models, corresponding to the affine algebra $\hat g$ at level $k = n - h^\vee$, where $h^\vee$ is the dual Coxeter number of $g$. Define

$$P_+^n \;:=\; \{\sum_{i=0}^n \lambda_i \omega^i \,|\, \lambda_i \in \mathbf{Z}_\geq \;,\; \sum_{i=0}^r a_i^\vee \lambda_i = n\,\} \;, \tag{3.4}$$

and $P^n_{++}$ similarly, except with $\mathbf{Z}_\geq$ replaced with $\mathbf{Z}_>$. The positive integers $a_i^\vee$ in eq.(3.4) are the *co-marks* [14] of $\hat g$. We read from eq.(3.1) that

$$S^{(n)}_{\lambda\sigma}/S^{(n)}_{\rho\sigma} \;=\; O^{(n)}_\lambda(\sigma)/O^{(n)}_\rho(\sigma) \;=:\; \chi^{(n)}_\lambda(\sigma) \;, \tag{3.5}$$

where

$$\chi_\lambda^{(n)}(\sigma) \; = \; \chi_\lambda(2\pi\,\sigma/n) \; . \tag{3.6}$$

This is the "happy fact" referred to in the introduction. $M$ and $L$ are lower triangular; therefore whenever $\lambda \in P_{++}^n$, this remarkable relation implies

$$\chi_\lambda^{(n)}(\sigma) \; = \sum_{\mu \in P_{++}^n} m_\lambda{}^\mu E_\mu^{(n)}(\sigma) \quad , \tag{3.7}$$

and

$$E_\lambda^{(n)}(\sigma) \; = \sum_{\mu \in P_{++}^n} \ell_\lambda{}^\mu \chi_\mu^{(n)}(\sigma) \quad . \tag{3.8}$$

Eq.(3.8) also holds for $\lambda \in P_{++} \cap P_+^n$. Using the unitarity of the modular $S$ matrix

$$\sum_{\nu \in P_{++}^n} O_\lambda^{(n)}(\nu)\, O_\mu^{(n)*}(\nu) \; = \sum_{\nu \in P_{++}^n} S_{\lambda\nu}^{(n)}\, S_{\nu\mu}^{(n)*} \; = \; \delta_\lambda^\mu \; , \tag{3.9}$$

we arrive at

$$\ell_\lambda{}^\mu \; = \sum_{\sigma \in P_{++}^n} E_\lambda^{(n)}(\sigma)\, O_\rho^{(n)}(\sigma)\, O_\mu^{(n)*}(\sigma) \; = \sum_{\sigma \in P_{++}^n} E_\lambda^{(n)}(\sigma)\, S_{\rho\sigma}^{(n)}\, S_{\mu\sigma}^{(n)*} \quad , \tag{3.10}$$

valid whenever both $\lambda \in P_{++} \cap P_+^n$ and $\mu \in P_{++}^n$. Eq.(3.10) can be generalized:

$$\sum_{\sigma \in P_{++}^n} \ell_\lambda{}^\sigma N_{\mu\sigma}^{(n)\nu} = \sum_{\sigma \in P_{++}^n} E_\lambda^{(n)}(\sigma)\, S_{\mu\sigma}^{(n)}\, S_{\nu\sigma}^{(n)*} \; , \tag{3.11}$$

where $N_{\mu\sigma}^{(n)\nu}$ are the WZW fusion rules, which we may take to be defined by Verlinde's formula [11]:

$$N_{\lambda\mu}^{(n)\quad\nu} \; = \sum_{\sigma \in P_{++}^n} \chi_\lambda^{(n)}(\sigma)\, S_{\mu\sigma}^{(n)}\, S_{\nu\sigma}^{(n)*} \quad . \tag{3.12}$$

Of course we also get directly from eq.(3.7) that

$$N_{\lambda\mu}^{(n)\nu} = \sum_{\sigma,\gamma \in P_{++}^n} m_\lambda{}^\gamma\, E_\gamma^{(n)}(\sigma)\, S_{\mu\sigma}^{(n)}\, S_{\nu\sigma}^{(n)*} \; . \tag{3.13}$$

Because $L$ is lower triangular, an easy argument gives

$$m_\lambda{}^\mu = (L^{(n)-1})_\lambda{}^\mu \tag{3.14}$$

for all $\lambda, \mu \in P_{++}^n$, where $L^{(n)}$ is defined to be the sublattice of $L$ obtained by restricting it to the set $P_{++}^n$. Thus equations like eq.(3.10) provide a simple method of calculating

dominant weight multiplicities $m_\lambda{}^\mu$. Moreover, if we find a permutation $\pi$ of $P^n_{++}$ which commutes with both $S^{(n)}$ and $E^{(n)}$, then it will be an exact symmetry of both $\ell$ and $m$. More generally, if $S^{(n)}$ and $E^{(n)}$ both transform "nicely" under a permutation $\pi$ of $P^n_{++}$, then we can expect to derive new relations for $\ell$ and $m$. This is the motivation for the following section.

## 4. New relations between multiplicities

In this section, we will show that symmetries of the Kac-Peterson modular matrices $S^{(n)}$ of WZW models give rise to new relations between finite-dimensional Lie algebra multiplicities.

The most obvious symmetry concerns the affine Weyl group $\widehat{W}$ of $\hat{g}$. We know [14] that it is isomorphic to the semi-direct product of the (finite) Weyl group $W$ with the group of translations in the coroot lattice $Q^\vee$. We also know that the $\widehat{W}$-orbit of any weight intersects $P^n_+$ in exactly one point. More precisely, let $\lambda \in M$ be some weight. Then there exists an element $\alpha$ in the coroot lattice of $g$, and some $w \in W$, such that

$$[\lambda] \ := \ w(\lambda + n\alpha) \ \in \ P^n_+ \ . \tag{4.1}$$

We will use this observation throughout this section. $[\lambda]$ is uniquely determined by $\lambda$ (and $n$), but $w$ will be only if $[\lambda] \in P^n_{++}$. Define $\epsilon(\lambda) := 0$ if $[\lambda] \notin P^n_{++}$, and $\epsilon(\lambda) := det(w)$ otherwise, where $w \in W$ satisfies eq.(4.1).

We read directly from eqs.(3.1),(3.2) respectively that

$$S^{(n)}_{\lambda\mu} = \epsilon(\lambda)\, S^{(n)}_{[\lambda]\mu} = \epsilon(\mu)\, S^{(n)}_{\lambda[\mu]} \ ; \tag{4.2}$$

$$E^{(n)}_\lambda(\sigma) = E^{(n)}_\lambda([\sigma]) = \frac{|W(\lambda-\rho)|}{|W([\lambda-\rho])|}\, E^{(n)}_{[\lambda-\rho]+\rho}(\sigma) \ . \tag{4.3}$$

By the argument which gave us eq.(3.10), we find that for any $\lambda \in P_{++}$, $\mu \in P^n_{++}$,

$$\sum_{\substack{\nu\in P_{++}\\ [\nu]=\mu}} \epsilon(\nu)\, \ell_\lambda{}^\nu = \sum_{\sigma\in P^n_{++}} E^{(n)}_\lambda(\sigma)\, S^{(n)}_{\rho\sigma}\, S^{(n)*}_{\mu\sigma} \ . \tag{4.4}$$

Thus for any $\mu \in P^n_{++}$, and any $\lambda \in P_{++}$ with $[\lambda-\rho]+\rho \in P^n_+$, we get the truncation

$$\ell_{[\lambda-\rho]+\rho}{}^\mu = \frac{|W([\lambda-\rho])|}{|W(\lambda-\rho)|} \sum_{\substack{\nu\in P_{++}\\ [\nu]=\mu}} \epsilon(\nu)\, \ell_\lambda{}^\nu \ . \tag{4.5}$$

Roughly speaking, eq.(4.5) says that if we know the $\ell$'s for "large" weights, then we know them for "small" ones. Incidently, if $[\lambda-\rho]+\rho\not\in P_+^n$, then (4.5) holds if we replace its LHS with a sum similar to that of its RHS. Similar comments hold below if $\pi_A(\lambda)\not\in P_+^n$ in (4.13), or $\pi_a(\lambda)\not\in P_+^n$ in (4.17).

Next, consider the symmetries involving the outer automorphisms of affine Lie algebras $\hat{g}$, or equivalently, the automorphisms of the *extended* Coxeter-Dynkin diagrams of $g$. If an outer automorphism is also a symmetry of the unextended Coxeter-Dynkin diagram of $g$, i.e. it fixes the extended node, then it is well known to be an exact symmetry $C$ (a *conjugation*) of both the $\ell$'s and $m$'s:

$$m_{C\lambda}^{\ C\mu}=m_\lambda^{\ \mu}\ ,\qquad \ell_{C\lambda}^{\ C\mu}=\ell_\lambda^{\ \mu}\ . \tag{4.6}$$

We are interested here instead in those automorphisms which are not conjugations. Denote such an automorphism by $A$, and the fundamental weights of $\hat{g}$ by $\omega^i$, with $i=0,1,2,\ldots,r$. There is one of these automorphisms $A=A_i$ for every node of the extended diagram with mark $a_i=1$; $A_i$ will send $\omega^0$ to $\omega^i$. Since

$$A(\lambda-n\omega^0)\ =\ w_A(\lambda-n\omega^0)\ , \tag{4.7}$$

with $w_A$ an element of the Weyl group $W$ of $g$, for all $\lambda\in P_+^n$, we have [15]

$$S^{(n)}_{A\lambda,\sigma}\ =\ S^{(n)}_{\lambda\sigma}\ \exp\left[-2\pi i\,(A\omega^0)\cdot\sigma\right]\ det(w_A)=S^{(n)}_{\lambda\sigma}\ \exp[-2\pi i\,(A\omega^0)\cdot(\sigma-\rho)]\ \ . \tag{4.8}$$

Similarly

$$E^{(n)}_\lambda(A\sigma)\ =E^{(n)}_\lambda(\sigma)\ \exp[-2\pi i\,(A\omega^0)\cdot(\lambda-\rho)]\ , \tag{4.9}$$

$$E^{(n)}_{A\lambda}(\sigma)\ =E^{(n)}_{\pi_A\lambda}(\sigma)\ \frac{|W(A\lambda-\rho)|}{|W(\pi_A(\lambda)-\rho)|}\ \exp\left[-2\pi i\,(A\omega^0)\cdot\sigma\right]\quad , \tag{4.10}$$

where $\pi_A$ denotes the one-to-one map from $P^n_{++}$ to $P_{++}$ given by

$$\pi_A(\lambda):=[\lambda-w_A^{-1}\rho]+\rho\ . \tag{4.11}$$

For fixed $g$, $w_A$ and hence the map $\pi_A$ is readily obtained from eq.(4.7) – $\pi_A$ will be the identity only when $A$ is. For example, for $g=su(r+1)$ and $A=A_j$ satisfying $A\omega^i=\omega^{i+j}$, we get

$$\left(w_A\lambda\right)_i\ =\ \begin{cases}\lambda_{i-j} & \text{if } i\neq j\\ -\lambda_1-\cdots-\lambda_r & \text{if } i=j\end{cases}\ , \tag{4.12}$$

$det(w_A) = (-1)^{rj}$, and $\pi_A(\lambda) = \lambda + (r+1)\,\omega^{r+1-j}$.

From our main result (3.10), we immediately find

$$\ell_{A\lambda}^{\ A\mu} \ = \ \ell_{\pi_A(\lambda)}^{\ \mu} \, \frac{|W(A\lambda-\rho)|}{|W(\pi_A\lambda-\rho)|} \, det(w_A) \ , \tag{4.13}$$

for any $\lambda, \mu \in P^n_{++}$, provided $\pi_A(\lambda) \in P^n_+$. Unfortunately $\pi_A$ will only be a permutation of $P^n_{++}$ in the trivial case when $A = id.$, so it is not easy to see what eq.(4.13) directly implies for the $m_\lambda^{\ \mu}$.

There are also Galois symmetries of the Kac-Peterson modular matrix $S^{(n)}$, first discovered in [16][17] (and generalized to all rational conformal field theories in [18]). The $S^{(n)}_{\lambda\mu}/F_n$ and $E^{(n)}_\lambda(\sigma)$ are polynomials with rational coefficients in a primitive $(nN)$-th root of unity, where $N = |M^*/M|^{\frac{1}{2}}$, $M$ here being the weight lattice of $g$. So, any polynomial relation involving them and rational numbers only, will also be satisfied if this primitive $(nN)$-th root of unity is replaced by another.

Let $a$ be an integer coprime to $nN$, and let $a(S^{(n)})$ denote the Kac-Peterson matrix after the primitive $(nN)$-th root of unity is replaced by its $a$-th power (ignoring here the irrelevant factor $F_n$). For such $a$, and for $\lambda \in P^n_{++}$, recall the quantities $[a\lambda] \in P^n_{++}$ and $\epsilon(a\lambda) \in \{\pm 1\}$ defined around eq.(4.1). For each $a$ coprime to $nN$, the map $\lambda \mapsto [a\lambda]$ is a permutation of $P^n_{++}$. From the form of the matrix $S^{(n)}$, it is easy to find

$$a\left(S^{(n)}_{\lambda\mu}\right) \ = \ \epsilon(a\lambda)\, S^{(n)}_{[a\lambda],\mu} \ = \ \epsilon(a\mu)\, S^{(n)}_{\lambda,[a\mu]} \ . \tag{4.14}$$

In a similar fashion, we find

$$a\left(E^{(n)}_\lambda(\sigma)\right) \ = \ \frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|} \, E^{(n)}_{\pi_a(\lambda)}(\sigma) \ = \ E^{(n)}_\lambda([a\sigma]) \ , \tag{4.15}$$

where $\pi_a$ denotes the one-to-one map from $P^n_{++}$ to $P_{++}$ defined by

$$\pi_a(\lambda) := [a\lambda - a\rho] + \rho \ . \tag{4.16}$$

A little work yields

$$\ell_\lambda^{\ \mu} \ = \ \epsilon(a\mu)\,\epsilon(a\rho)\, \frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|} \sum_{\nu\in P^n_{++}} \ell_{\pi_a(\lambda)}^{\ \nu}\, N^{(n)\ \ [a\mu]}_{\nu,[a\rho]} \ , \tag{4.17}$$

whenever $\mu \in P^n_{++}$ and $\lambda, \pi_a(\lambda) \in P_{++} \cap P^n_+$. Here we have used the Verlinde formula [11] (3.12) for the fusion coefficients $N^{(n)\ \nu}_{\lambda\mu}$. Let $N^{(n)}_\lambda$ denote the *fusion matrix* defined by

$(N^{(n)}_\lambda)^\nu_\mu := N^{(n)\ \nu}_{\lambda\mu}$. The matrix $N^{(n)}_{[a\rho]}$ will always be invertible [18], so eq.(4.17) tells us that for any fixed $\lambda \in P_{++}$, the values $\ell_{\pi_a(\lambda)}^{\ \ -}$ will be known once the $\ell_\lambda^{\ -}$ are known, and conversely, provided $a$ and $n$ satisfy the usual conditions. Eq.(4.17) can also be interpreted as an expression for the fusion matrices $N^{(n)}_{[a\rho]}$ in terms of the $\ell$'s and $m$'s.

It is again difficult to express this Galois symmetry directly at the level of the multiplicities $m_\lambda^{\ \mu}$ themselves. But if $\pi_a$ *is* a permutation of $P^n_{++}$, we get from eq.(4.17):

$$m_\lambda^{\ \mu} = \ \epsilon(a\mu)\,\epsilon(a\rho)\ \frac{|W(\pi_a\lambda-\rho)|}{|W(\lambda-\rho)|}\sum_{\nu\in P^n_{++}}(N^{(n)-1}_{[a\rho]})^\nu{}_{[a\lambda]}\, m_\nu^{\ \pi_a(\mu)}\ . \tag{4.18}$$

A special case of eq.(4.17) occurs when $[a\rho]=\rho$. (More generally, a similar simplification happens whenever $[a\rho]=A\rho$ for some outer automorphism $A$.) Then $\epsilon(a\rho)=\epsilon(a\lambda)$ for all $\lambda\in P^n_{++}$ (apply eq.(4.14) with $\mu=\rho$, together with the fact that $S^{(n)}_{\rho\nu}>0$ for all $\nu\in P^n_{++}$). Eq.(4.17) reduces to

$$\ell_\lambda^{\ \mu} = \frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|}\,\ell_{\pi_a(\lambda)}^{\ \ [a\mu]}\ . \tag{4.19}$$

For example this happens whenever $a=-1$, and we get an example of eq.(4.6).

Similarly, suppose $[a\mu] = A\rho$, for some outer automorphism $A$. Then provided $\lambda,\pi_a(\lambda)\in P_{++}\cap P^n_+$, eq.(4.17) reduces to

$$\ell_\lambda^{\ \mu} = \epsilon(a\mu)\,\epsilon(a\rho)\,\frac{|W(\lambda-\rho)|}{|W(\pi_a\lambda-\rho)|}\,\ell_{\pi_a\lambda}^{\ \ A[a\rho]}\ . \tag{4.20}$$

Another noteworthy special case of eq.(4.17) involves those weights $\lambda'$ with the property that $\ell_{\lambda'}^{\ \mu}=\delta^\mu_{\lambda'}$ for all $\mu\in P_{++}$, i.e. those $\lambda'$ for which $\lambda'-\rho$ is a miniscule weight. For $su(r+1)$, they are the fundamental weights $\lambda'=\omega^i+\rho$. Then for any $\lambda\in P^{(n)}_{++}$ with $\pi_a(\lambda)=\lambda'$,

$$\ell_\lambda^{\ \mu} = \epsilon(a\mu)\,\epsilon(a\rho)\,\frac{|W(\lambda-\rho)|}{|W(\lambda'-\rho)|}\,N^{(n)\ \ [a\mu]}_{\lambda',[a\rho]}\ , \tag{4.21}$$

if, as usual, $a$ is coprime to $nN$ and $\lambda',\mu\in P^n_{++}$. The fusions involving the fundamental weights $\omega^i+\rho$ are easy to compute, so the RHS of eq.(4.21) can be explicitly evaluated in all cases. One of the reasons equations (4.17)-(4.21) could be interesting is because they suggest the rank-level duality that WZW fusions satisfy [19] could appear in some way in the $\ell$'s and $m$'s.

## 5. Examples: the case of $g = su(3)$

For concrete illustrations of our results, we will focus on the example of $g = su(3)$. The comments in this section can be extended to any $su(r+1)$ without difficulty.

Consider first eq.(4.13). It reduces to

$$\ell_{(\lambda_1,\lambda_2)}^{(\mu_1,\mu_2)} = \ell_{(n-\lambda_1-\lambda_2+3,\lambda_2)}^{(n-\mu_1-\mu_2,\mu_2)} , \qquad \forall n \in \mathbf{Z}_> , \tag{5.1}$$

provided only that: $\mu_1 + \mu_2 < n$; $\lambda_2, \mu_1, \mu_2 \geq 1$; and either $\lambda_1 + \lambda_2 < n$, $\lambda_1 \geq 3$, or $\lambda_1 + \lambda_2 \leq n$, $\lambda_1 > 3$.

Eq.(5.1) is very powerful. For example, suppose $\lambda_1 \geq 3$. Put $n = \lambda_1 + \lambda_2 + 1$. Then eq.(5.1) reduces $\ell_\lambda^{\ \mu}$ to $\ell_{\lambda'}^{\ \mu'}$, where $\lambda' = (4, \lambda_2)$. In fact, if both $\lambda_1, \lambda_2 \geq 3$ and $\mu \in P_{++}$, we obtain from eq.(5.1) (once we compute the values of $\ell_{(4,4)}^{\ -}$):

$$\ell_\lambda^{\ \mu} = \begin{cases} +1 & \text{if } \mu \in \{\lambda,\ (\lambda_1-3,\lambda_2),\ (\lambda_1,\lambda_2-3)\} \\ -1 & \text{if } \mu \in \{(\lambda_1-2,\lambda_2+1),\ (\lambda_1+1,\lambda_2-2),\ (\lambda_1-2,\lambda_2-2)\} \\ 0 & \text{otherwise} \end{cases} . \tag{5.2}$$

In fact, using eq.(4.5), we find that eq.(5.2) gives the correct value of $\ell_\lambda^{\ \mu}$ for any $\lambda, \mu \in P_{++}$ (whether or not $\lambda_1 \geq 3$), with one exception: $\ell_{(2,2)}^{(1,1)} = -2 \neq 0$,

Next, let us turn to the Galois symmetries. It is not difficult to see that $[a\rho] = \rho$ only for $a \equiv \pm 1 \pmod{n}$, in which case $\pi_a(\lambda) = \pi_A(\lambda)$ for some outer automorphism $A$. Thus for $g = su(3)$, eq.(4.19) only gives information also obtainable from eq.(5.1).

However, eqs.(4.17),(4.20),(4.21) are very strong here. To give one explicit example, consider $\lambda = (4,4)$, $a = 5$, $n = 8$. Then $\pi_a\lambda = (2,2)$. The only possible nonzero elements of $\ell_{(2,2)}^{\ -}$ are $\ell_{(2,2)}^{(2,2)} = 1$ and $\ell_{(2,2)}^{(1,1)}$. We get from eq.(4.17) that e.g.

$$\begin{aligned} &\ell_{(4,4)}^{(3,3)} = 0 , \quad \ell_{(4,4)}^{(1,1)} = N_{(2,2),\,(3,3)}^{(8)\ \ (3,3)} + \ell_{(2,2)}^{(1,1)} , \\ &\ell_{(4,4)}^{(2,2)} = -N_{(2,2),\,(3,3)}^{(8)\ \ (2,2)} , \quad \ell_{(4,4)}^{(5,2)} = -N_{(2,2),\,(3,3)}^{(8)\ \ (2,5)} , \quad \ell_{(4,4)}^{(1,4)} = N_{(2,2),\,(3,3)}^{(8)\ \ (1,4)} . \end{aligned} \tag{5.3}$$

Note that the relations (4.5), (4.13), and (4.17), together with the selection rule "$\ell_\lambda^{\ \mu} \neq 0$ requires $\lambda - \mu \in \mathbf{Z}_\geq\{\alpha_1, \ldots, \alpha_r\}$", and the normalization $\ell_\lambda^{\ \lambda} = 1$, easily determine all values of $\ell_\lambda^{\ \mu}$ and hence $m_\lambda^{\ \mu}$. In particular, we have seen that eqs.(4.5) and (4.13) determine all $\ell_\lambda^{\ \mu}$ provided the values $\ell_{(4,4)}^{\ -}$ are known. Eq.(5.3) fixes all values of $\ell_{(4,4)}^{\ -}$ except $\ell_{(4,4)}^{(1,1)}$, but by eq.(5.1) we find $\ell_{(4,4)}^{(1,1)} = \ell_{(4,4)}^{(7,1)} = 0$. A similar conclusion should apply to any $su(r+1)$.

## 6. Conclusion

Our main result is the expression (3.10) for the inverse of the matrix of dominant weight multiplicities, in terms of the Kac-Peterson modular matrix $S^{(n)}$ and the even Weyl orbit sums $E_\lambda^{(n)}(\sigma)$. Surely the even Weyl orbit sums will find other uses in the study of WZW models and affine Kac-Moody algebras.

Also obtained were relations (4.5), (4.13) and (4.17) among the "inverse multiplicities" ${\ell_\lambda}^\mu$, that are consequences of the symmetries of the Kac-Peterson matrices and the $E_\lambda^{(n)}(\sigma)$. Relations between the dominant weight multiplicities ${m_\lambda}^\mu$ follow in certain cases, as eq. (4.18) shows. These relations would be difficult to understand from the point of view of Lie groups and their semi-simple Lie algebras only, but are quite natural in WZW models, or in their affine Kac-Moody current algebras. The new relations arise because the semi-simple symmetry algebra of a WZW model is a special subalgebra of its affine current algebra. The affine Weyl and outer automorphism symmetries are especially natural in the Kac-Moody context. Perhaps the Galois symmetries can be better understood in terms of affine Kac-Moody algebras.

The debt we owe to the previous work of Patera and Sharp [12] is obvious when comparing sections 2 and 3. We should also mention a paper by Moody and Patera [20]. In it class functions of an arbitrary semi-simple compact Lie group are decomposed into sums of irreducible characters. The method uses the characters of elements of finite order (EFO's) of the group to approximate the characters of arbitrary elements. These characters of EFO's have a marked similarity to the Kac-Peterson ratios $\chi_\lambda^{(n)}(\sigma)$. Moody and Patera also use modular arithmetic to ensure that the approximation leads to the correct answers, if a suitable set of EFO's is chosen. In a similar way, we automatically recover the exact weight multiplicities from the Kac-Peterson ratios, which are the Lie algebra characters evaluated at special points.

Further comparison with [20] is clearly warranted. Perhaps it will help us toward a better understanding of WZW models, one that approaches the current understanding of semi-simple compact Lie groups and their algebras.

## Acknowledgements

T.G. and M.W. thank the IHES for hospitality, and M.W. also thanks R.T. Sharp and P. Mathieu for discussions.

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